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LETTER TO THE EDITOR

Algebraic aspects of the descent equations

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Abstract. The algebraic meanings of the descent equations, $\delta\omega_4^1 = \mathrm{d}\omega_3^2$ and $\delta\omega_3^2 = \mathrm{d}\omega_2^3 \equiv \omega_3^3$, are examined in the Hamiltonian framework.

It is well understood that the descent procedure [1–5] plays an important role in deriving the anomalies in odd and/or even dimensions. In the four-dimensional case, we have the following descent equations:

$$\ldots$$

$$\delta\omega_4^1 = \mathrm{d}\omega_3^2 \tag{1}$$

$$\delta\omega_3^2 = \mathrm{d}\omega_2^3 \equiv \omega_3^3 \tag{2}$$

$$\ldots$$

where the convention is the same as in Zumino [4]. In this letter I examine the algebraic meanings of (1) and (2) and show that they can be derived from the Jacobi and the fourfold constraint identities for the Hamiltonian and the Gauss law constraints, respectively.

Following [6, 7] the canonical equations of motion of gauge fields $(A_i^a(x))$ interacting with sources $(J^a(x))$ are

$$\partial_0 E_i^a(x) = [E_i^a(x), H] = J_i^a(x) + (D_j F^{ji}(x))^a \tag{3}$$

$$E_i^a(x) = -\partial_0 A_i^a(x) = -[A_i^a(x), H] \tag{4}$$

where H is the Hamiltonian. It is easily shown from (3) and (4) that the Gauss law constraint

$$G^a(x) = J_0^a(x) + (D_i E^i(x))^a$$

satisfies the anomalous Ward identity

$$\partial_0 G(u) = [G(u), H] = \sigma(E, A; u) \tag{5}$$

$$G(u) = \int_{\mathbb{R}^3} \mathrm{d}^3x\, u^a(x) G^a(x) \qquad u = t^a u^a \tag{6}$$

$$\sigma(E, A; u) = \int_{\mathbb{R}^3} \mathrm{d}^3x\, u^a(x) \sigma^a(x) \tag{7}$$

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and $\sigma^a(x) = (D_\mu J^\mu(x))^a$ is the consistent anomaly in Weyl gauge; $E = t^a E_i^a \mathrm{d}x^i$ and $A = t^a A_i^a \mathrm{d}x^i$ are the differential forms.

It is natural to consider the Jacobi identity for (5) as

$$\begin{aligned} 0 &= [G(u), [G(v), H]] + [G(v), [H, G(u)]] + [H, [G(u), G(v)]] \\ &= [G(u), \sigma(E, A; v)] - [G(v), \sigma(E, A; u)] + [H, [G(u), G(v)]]. \end{aligned} \tag{8}$$

Using the commutator, $[G(u), G(v)] = G([u, v]) + W(A; u, v)$, we obtain

$$\partial_0 W(A; u, v) = [G(u), \sigma(E, A; v)] - [G(v), \sigma(E, A; u)] - \sigma(E, A; [u, v]). \tag{9}$$

It can be shown [6] that the RHS of (10) is the BRST transformation of the consistent anomaly

$$(\delta\sigma)(E, A; u, v) \equiv [G(u), \sigma(E, A; v)] - [G(v), \sigma(E, A; u)] - \sigma(E, A; [u, v]). \tag{10}$$

Thus (9) becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \omega_3^2(A; u, v) &\equiv \frac{\mathrm{d}}{\mathrm{d}t} W(A; u, v) = \partial_0 W(A; u, v) \\ &= (\delta\sigma)(E, A; u, v) \equiv (\delta\omega_4^1)(E, A; u, v) \end{aligned} \tag{11}$$

which is consistent with (1). It follows that the descent equation (1) is nothing but the Jacobi identity (8).

In order to obtain the meaning of (2), we consider the fourfold constraint identity for (5); it is [8]

$$\begin{aligned} &[G(u), [G(v), [G(w), H]] + \text{perm}] + \text{perm} \\ &\quad = [G([u, v]), [G(w), H]] + \text{perm} - [G([v, w]), [H, G(u)]] + \text{perm} \\ &\quad\quad + [G([u, w]), [H, G(v)]] + \text{perm}. \end{aligned} \tag{12}$$

Now suppose that

$$[G(u), [G(v), G(w)]] + \text{perm} = Z(A; u, v, w) \tag{13}$$

where $Z(A; u, v, w)$ is the anomalous Jacobian. Then (12) becomes

$$\begin{aligned} Z(A; u, v, w) &= [G(u), W(A; v, w)] + [G(v), W(A; w, u)] + [G(w), W(A; u, v)] \\ &\quad + W(A; u, [v, w]) + W(A; v, [w, u]) + W(A; w, [u, v]). \end{aligned} \tag{14}$$

The RHS of (14) is just the BRST transformation of the commutator anomaly $W(A; u, v)$. Therefore from (14) we have

$$\begin{aligned} \omega_3^3(A; u, v, w) &\equiv Z(A; u, v, w) = (\delta W)(A; u, v, w) \\ &\equiv (\delta\omega_3^2)(A; u, v, w) \end{aligned} \tag{15}$$

which coincides with (2). Thus we can conclude that the descent equation (2) is nothing but the fourfold constraint identity (12).

In conclusion, we have obtained the clear meanings of the descent equations (1) and (2). If we are given an expression of the consistent anomaly, then from (9) and (14) we can easily calculate the commutator anomaly W and the anomalous Jacobian Z[6]. Therefore, the present analysis should throw further light on understanding anomalous gauge theories.

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